A Condition Number Scaling Policy for Stability Robustness Analysis

The condition number of the process transfer function $G(s)$ has been proposed and used as an indicator of the inherent controllability of a process even though it is a scale-dependent quantity. A rapid suboptimal scaling policy known as G-balancing is developed for the case of the l_2 norm where the condition number is the ratio of the largest to the smallest singular value of $G(s)$. For 2×2 systems, G-balancing gives the same optimal scaling as an available analytical solution. The higher order examples given compare the results of G-balancing with the known analytic results for l_2 norm optimal scaling.

T. Chanh Nguyen,
Geoffrey W. Barton,
John D. Perkins
Department of Chemical Engineering
University of Sydney
N.S.W. 2006, Australia

Richard D. Johnston
School of Chemical Engineering
and Industrial Chemistry
University of New South Wales
Kensington, N.S.W. 2033, Australia

Introduction

To design chemical processes that are amenable to high-quality control, it is imperative that control considerations enter all stages of plant design. Recent research (Johnston and Barton, 1985; Lau et al., 1985; Doyle and Stein, 1981) has shown singular value analysis (SVA) to be a useful design tool in this regard.

Morari and coworkers (Holt and Morari, 1985; Grossmann and Morari, 1983; Morari, 1982) and Johnston and Barton (1985) have shown that SVA may be used to identify characteristics of the process design itself, as distinct from the control system design, which limit the inherent controllability of the system. Two quantities, the minimum singular value and the condition number of the process transfer function matrix $G(s)$ have proved to be most useful in assessing the controllability of alternative plant designs.

Processes with a large minimum singular value of $G(s)$ over a wide range of frequencies are less susceptible to manipulated variable saturation as a result of control action than are processes with a small minimum singular value. The process condition number, on the other hand, provides a measure of the sensitivity of the system to model uncertainty (model/plant mismatch). Processes with a small condition number over a wide frequency range are more tolerant of model/plant mismatch than those with a large condition number.

However, singular values and the condition number depend on the scaling or the physical dimensions of the system variables. Unless an appropriate scaling policy is adopted, the results of the analysis will have little physical significance, reflecting numerical features of the process model rather than the true process characteristics. This paper proposes a new, simple to apply, scaling policy for use in singular value analysis.

Singular Value Analysis

Norm definitions

Singular value analysis of a process is based on vector and matrix norms. Three vector norms are commonly employed in numerical analysis. They are defined as

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p} \quad p = (1, 2, \infty) \tag{1}$$

and are called the l_1, l_2, and l_∞ norms. The latter may be interpreted as

$$\|x\|_\infty = \max_i |x_i| \quad i = 1, 2, \ldots, m \tag{2}$$

The matrix norms subordinate to these vector norms are denoted by $\|A\|_p$, where A is a general matrix, and are defined as follows:

$$\|A\|_1 = \max_j \sum_i |a_{ij}| \tag{3}$$

$$\|A\|_\infty = \max_i \sum_j |a_{ij}| \tag{4}$$

Correspondence concerning this paper should be addressed to G.W. Barton.

$$\|A\|_2 = \max_i \lambda_i^{1/2} (A^*A) \tag{5}$$

where a_{ij} is an individual element of A, λ_i $(i = 1, 2, \ldots, m)$ are the eigenvalues, and A^* is the complex conjugate transpose of A. Furthermore, the condition number of a square matrix A with respect to the p-norm may be defined as

$$\text{cond}_p (A) = \|A\|_p \|A^{-1}\|_p \tag{6}$$

Singular value analysis makes use of l_2 vector and matrix norms. The l_2 vector norm is commonly known as the Euclidean vector norm, while the l_2 matrix norm is the maximum singular value, denoted by $\sigma^*\{A\}$. The singular values of a matrix A are defined as the positive square roots of the eigenvalues of the Hermitian matrix A^*A. The maximum singular value corresponds to the largest eigenvalue of this matrix while the minimum singular value, denoted by $\sigma_*\{A\}$, is derived from the smallest eigenvalue.

Scaling

The dynamic behavior of a chemical process may be approximated by the linear input-output relationship

$$y(s) = G(s)\, u(s) \tag{7}$$

where

$y(s)$ = the m-dimensional vector of process outputs

$u(s)$ = the m-dimensional vector of process inputs

$G(s)$ = the mxm process transfer function matrix

Taking 2-norms of both sides of Eq. 7 yields

$$\frac{\|y\|_2}{\|u\|_2} \le \sigma^*\{G\} \tag{8}$$

Furthermore, MacFarlane and Scott-Jones (1979) have shown that

$$\sigma_*\{G\} \le \frac{\|y\|_2}{\|u\|_2} \le \sigma^*\{G\} \tag{9}$$

and it follows from Eqs. 5 and 6 that the process condition number subordinate to the l_2 norm (hereafter denoted by γ) is given by

$$\gamma = \frac{\sigma^*\{G\}}{\sigma_*\{G\}} \tag{10}$$

As outlined in the first section, it is this quantity that has found wide acceptance as a criterion for assessing plant controllability. However, its numerical value depends upon the scaling of the process input and output variables. In other words, the condition number of the process model can be altered by pre- and post-multiplying the system by real diagonal matrices J and F, Figure 1, to give the scaled matrix

$$\hat{G} = JGF \tag{11}$$

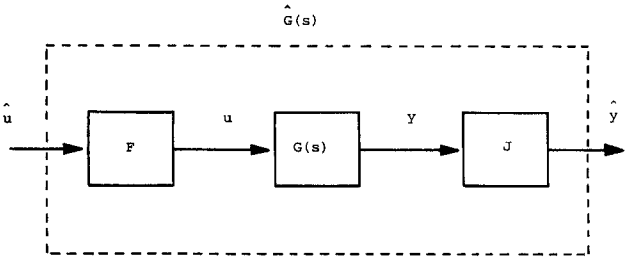

Figure 1. Scaling of process inputs and outputs.

and the scaled relationship

$$\hat{y} = \hat{G}\, \hat{u} \tag{12}$$

Thus, the relationships between the unscaled vectors y and u and their scaled counterparts $\hat{y}$ and $\hat{u}$ are

$$\hat{y} = J\, y \tag{13}$$

and

$$\hat{u} = F^{-1}\, u \tag{14}$$

Mathematicians have developed a number of scaling methods to facilitate the solution of systems of equations (Forsythe and Moler, 1967; Tomlin, 1975; Gill et al., 1981). Lau and Jensen (1985) have briefly reviewed and tested a number of these empirical and semiempirical scaling techniques. As an example, the equilibration method (Forsythe and Moler, 1967) is aimed at making the maximum entry in each row and in each column of the matrix the same magnitude. Alternatively, the geometric scaling method (Gill et al., 1981) scales each row and column by the geometric mean of its largest and smallest elements. Both techniques have found wide application, although the results are often dependent on the order of scaling, namely row scaling followed by column scaling or vice versa.

The above scaling methods are aimed at making the solution method for a set of equations more robust. However, when assessing the controllability of a process on the basis of the condition number of $G(s)$, the problem is not one of solving the set of Eqs. 7. To make a comparison of alternative process designs meaningful, the scaling of the input and output variables must be consistent. Perkins and Wong (1985) advocated that optimal scaling be employed. They determined the scaling matrices F and J that minimized the l_∞ norm condition number, at each frequency considered. Analytical techniques are available to find such optimal scaling matrices for the l_1 and l_∞ norms but not for the l_2 norm. Wong (1985) gives a comprehensive review of the results for all three norms while some of the relevant l_2 norm results are given next.

Although no analytical solution is available to provide the scaling matrices F and J that minimize γ, there are, however, some results available as a guide to the "optimality" of any scaling employed. The pertinent theorems are given here.

Theorem 1 (Bauer, 1963). An upper bound on the condition number of an optimally scaled matrix $\hat{G} = JGF$ is given by

$$\min_{J,F} \gamma(JGF) \le \pi(|G|\,|G^{-1}|) \tag{15}$$

where $|G|$ represents the matrix G with the modulus taken of each of its elements and $\pi(A)$ represents the Perron root of the matrix A, defined as

$$\pi(A) = \max_i |\lambda_i| \quad i = 1, 2, \ldots, m \tag{16}$$

Bauer (1963) gives the scaling matrices J and F that lead to this upper bound. Thus, any "optimal" scaling policy should achieve at least the condition number given by the upper bound which is, in fact, the optimal condition number in the l_∞ norm. Further, if G is a checkerboard matrix (as defined in Bauer's paper), Eq. 15 becomes an equality. Thus, it is unnecessary to devise near-optimal scaling policies for this class of matrix.

Theorem 2 (Golub and Varah, 1974). In the singular value decomposition, $G = U\Sigma V^T$ (Stewart, 1973), if u_{ij} and v_{ij} denote the elements of the ith row and jth column of the unitary matrices U and V, respectively, then G is optimally scaled subordinate to the l_2 norm if and only if

$$|u_{i1}| = |u_{im}| \text{ and } |v_{i1}| = |v_{im}| \quad (i = 1, 2, \ldots, m) \tag{17}$$

This theorem says that a matrix is optimally scaled if the first and last columns of U and V have components of equal magnitude. This is known as the EMC property and can be used as a check on the "optimality" of any scaling employed.

Theorem 3. For a matrix G to have a condition number, γ, of unity, the following two conditions must be satisfied by the row and column vectors comprising G:

1. All row and column vectors must have the same Euclidean norm.
2. Both the row and column vectors form orthogonal sets.

The proof of this theorem is given in the appendix.

G-Balancing

The problem of finding the diagonal matrices J and F that minimize the condition number of $\hat{G}$ may, of course, be solved by an optimization method. However, if the dimensions of G are mxm, $2m$ optimization variables must be solved for. Furthermore, it is often necessary to minimize the condition number over a wide range of frequencies. In these cases, the size of the optimization problem will grow considerably since the matrices J and F will need to be determined at a number of frequencies over this range. While the optimization approach will yield the best scaling matrices, a rapid suboptimal scaling method would prove increasingly valuable as the size of the problem increased. The G-balancing technique described below is such a technique.

Development of the methodology

The objective of G-balancing is to try to make the 2-norms of all the rows and columns of the scaled matrix unity. The justification for this approach is twofold. First, we have seen from theorem 3 above that one condition for a matrix to have unit condition number is that its row and column vectors all have the same 2-norm. Second, as we shall discuss below, by scaling the transfer function matrix in this way, the variation of the gain of the system as the input direction is changed is reduced. Since the range of the gain is bounded by the minimum and maximum singular values, it is to be expected that the difference between these singular values, and hence the condition number, is thereby reduced.

G-balancing consists of three steps:

1. Normalization of the input and output variables by dividing by their steady-state values
2. Column balancing
3. Row balancing

Each of these three operations is discussed below.

Normalization

The input and output variables are normalized by dividing each by its steady-state value. This usually reduces the variation in magnitude of the entries in $G(s)$ and gives uniform meaning to each variable, namely a fractional change from the steady-state value.

Column balancing

The objective of column balancing is to scale the inputs such that there is equal gain between the output vector y and the individual inputs u_i $(i = 1, 2, \ldots, m)$. This provides a partial fulfilment of condition 1 of theorem 3, i.e., that all the column vectors have the same Euclidean norm.

The input-output relationship, Eq. 7, may be written as:

$$y = [c_1 c_2 \ldots c_m]u \tag{18}$$

where $c_1, c_2, \ldots, c_m$ are m-dimensional column vectors.

Equation 18 may be rewritten as

$$y = c_1 u_1 + c_2 u_2 + \cdots + c_m u_m \tag{19}$$

If input u_i changes and all other inputs remain at zero then

$$y = c_i u_i \tag{20}$$

and

$$\frac{\|y\|}{|u_i|} = \|c_i\| \tag{21}$$

If u_i is scaled according to

$$\hat{u}_i = \|c_i\| u_i \tag{22}$$

then

$$\frac{\|y\|}{|\hat{u}_i|} = 1 \quad \text{for all} \quad i = 1, 2, \ldots, m \tag{23}$$

In other words, by using the Euclidean norm of the column vectors c_i as scaling factors for the inputs u_i, there is equal gain between the vector y and each of the individual scaled inputs $\hat{u}_i$. When this is the case, the columns of $G(s)$ are said to be balanced.

Row balancing

Another way of affecting the variability of the gain of the system is to influence the range of sensitivity of each individual output to a general input, i.e. to influence $|y_i|/\|u\|$ for each i $(i = 1, 2, \ldots, m)$. This can be achieved by row balancing as follows.

Equation 7 may be written as

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix} = \begin{bmatrix} r_1^T \\ r_2^T \\ \cdot \\ \cdot \\ \cdot \\ r_m^T \end{bmatrix} u \tag{24}$$

where r_j^T $(j = 1, 2, \ldots, m)$ are row vectors. Taking norms of both sides of Eq. 24 gives:

$$\begin{bmatrix} |y_1| \\ |y_2| \\ \cdot \\ \cdot \\ \cdot \\ |y_m| \end{bmatrix} \leq \begin{bmatrix} \|r_1^T\| \\ \|r_2^T\| \\ \cdot \\ \cdot \\ \cdot \\ \|r_m^T\| \end{bmatrix} \|u\| \tag{25}$$

The sensitivity of an individual output y_j to all inputs u is therefore measured by the Euclidean norm of the row vector r_j^T. By scaling the outputs according to:

$$\hat{y}_j = y_j / \|r_j^T\| \tag{26}$$

then

$$\begin{bmatrix} |\hat{y}_1|/\|u\| \\ |\hat{y}_2|/\|u\| \\ \vdots \\ |\hat{y}_m|/\|u\| \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{27}$$

When the outputs are scaled according to Eq. 26, the rows of $G(s)$ are said to be balanced.

A combination of normalization and column and row balancing has proved to be a reliable technique for scaling a process model such that the condition number approaches the minimum possible with diagonal scaling matrices. When all columns (rows) of the matrix have equal magnitude (i.e., Euclidean norms), the columns (rows) are said to be perfectly balanced. When both the rows and columns are perfectly balanced, the matrix is said to be perfectly balanced. In the light of the above discussion, the scaling matrices J and F are, thus,

$$J = \begin{bmatrix} 1/\|r_1^T\| & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/\|r_m^T\| \end{bmatrix} \tag{28}$$

and

$$F = \begin{bmatrix} 1/\|c_1\| & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/\|c_m\| \end{bmatrix} \tag{29}$$

Note that after the row balancing the columns may no longer be perfectly balanced. Therefore, column balancing followed by row balancing (which represents one balancing cycle) may be repeated as often as necessary, say k times, in order to bring the matrix to a perfectly balanced state, giving

$$\hat{G} = J_k J_{k-1} \ldots J_1 G F_1 F_2 \ldots F_k$$

As the perfectly balanced state of the $\hat{G}$ matrix is approached, both J_k and F_k approach the identity matrix. All row and column vectors in the perfectly balanced state have Euclidean norms of one. In the examples studied to date, convergence is rapid, with generally only two or three balancing cycles required to bring the matrix close to a perfectly balanced state. While the resultant matrix at an intermediate balancing cycle changes with the order of balancing (i.e., row followed by column balancing or vice versa), the final perfectly balanced state of $\hat{G}$ remains the same. In other words, the perfectly balanced state is independent of the balancing order.

Simultaneous balancing

For a two-input/two-output system, simultaneous row and column balancing may be achieved analytically. Let

$$G = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{30}$$

$$J = \begin{bmatrix} j_1 & 0 \\ 0 & j_2 \end{bmatrix} \tag{31}$$

and

$$F = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} \tag{33}$$

Then

$$\hat{G} = JGF = \begin{bmatrix} af_1j_1 & bf_2j_1 \\ cf_1j_2 & df_2j_2 \end{bmatrix} \tag{33}$$

$\hat{G}$ will be perfectly balanced when

$$(af_1j_1)^2 + (bf_2j_1)^2 = (cf_1j_2)^2 + (df_2j_2)^2 \quad \text{(row)} \tag{34}$$

and

$$(af_1j_1)^2 + (cf_1j_2)^2 = (bf_2j_1)^2 + (df_2j_2)^2 \quad \text{(column)} \tag{35}$$

both of which are satisfied when

$$|ab|j_1^2 = |cd|j_2^2 \tag{36}$$

$$|a/b|f_1^2 = |d/c|f_2^2 \tag{37}$$

By arbitrarily choosing one of the entries in each of J and F, the other may be determined from Eqs. 36 and 37.

Grosdidier et al. (1985) have shown that for a 2×2 matrix the optimum pre- and post-scaling matrices J and F may be derived analytically. Their result is identical to Eqs. 36 and 37, showing that G-balancing gives the optimal scaling matrices for 2×2 systems.

As an example of simultaneous balancing, consider the system described by the steady-state gain matrix

$$G(0) = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \tag{38}$$

Equations 36 and 37 become

$$j_1^2 = (5/6)\, j_2^2$$
$$f_1^2 = (15/2)\, f_2^2$$

and arbitrarily choosing $j_1 = f_1 = 1$ gives $j_2 = 1.095$, $f_2 = 0.365$. The $\hat{G}(0)$ matrix after simultaneous balancing is therefore

$$\hat{G}(0) = JGF = \begin{bmatrix} 2 & 1.095 \\ -1.095 & 2 \end{bmatrix} \tag{39}$$

which has the minimum possible condition number of unity.

With the exception of 2×2 systems, it is not claimed that G-balancing will necessarily find the optimum scaling matrices. However, we have found on a large number of examples that G-balancing provides a rapid and simple means of reducing the l_2 norm condition number of a matrix and, as the following examples show, the result is often very close to the optimum.

Examples of G-Balancing

In the examples that follow, matrices of real numbers will be used. However, it should be noted that the G-balancing scaling technique is particularly suitable for control system analysis where the entries in the process transfer function matrix $G(s)$ may be represented as complex numbers that are a function of frequency. The G-balancing method may be applied to reduce the condition number at any frequency, not just at steady state.

Example 1

The first example to be considered is

$$G_1 = \begin{bmatrix} 12 & 2 & 60 \\ 1 & 2 & 8 \\ 35 & 100 & 10 \end{bmatrix} \tag{40}$$

which has a condition number of $\gamma = 98.6961$. Table 1 shows the rapidity with which G-balancing reduces the condition number toward the ultimate value at convergence ($\gamma_{min} = 5.5412$). The scaling matrices J and F converge to

$$J = \text{diag}\,(1.5179, 11.7447, 0.3828) \tag{41}$$

and

$$F = \text{diag}\,(0.0392, 0.0222, 0.0076) \tag{42}$$

Table 1. Reduction of Condition Number with Balancing Cycles for Example 1

No. of Cycles	Condition No., γ
0	98.6961
1	12.1515
2	7.6261
3	6.0509
5	5.5550
10	5.5412
15	5.5412

giving

$$\hat{G} = JGF = \begin{bmatrix} 0.7149 & 0.0674 & 0.6959 \\ 0.4610 & 0.5216 & 0.7179 \\ 0.5259 & 0.8501 & 0.0293 \end{bmatrix} \tag{43}$$

The l_∞ matrix norm for this example is given by

$$\pi(|G_1||G_1^{-1}|) = 5.6647 \tag{44}$$

showing that G-balancing has reduced the condition number below the upper bound given by theorem 1. The singular value decomposition of $\hat{G}$ is

$$U = \begin{bmatrix} 0.5740 & -0.5929 & -0.5648 \\ 0.6331 & -0.1161 & 0.7653 \\ 0.5193 & 0.7969 & -0.3087 \end{bmatrix} \tag{45}$$

$$\Sigma = \text{diag}\,(1.5373, 0.7481, 0.2774) \tag{46}$$

and

$$V = \begin{bmatrix} 0.6343 & -0.0784 & -0.7691 \\ 0.5274 & 0.7713 & 0.3564 \\ 0.5652 & -0.6317 & 0.5306 \end{bmatrix} \tag{47}$$

The above decomposition, together with theorem 2, shows that G-balancing does not give an optimally scaled matrix.

Example 2

This example is one considered by Golub and Varah (1974). The matrix to be scaled is

$$G_2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{48}$$

for which $\gamma = 27.3727$. Golub and Varah gave the optimally scaled matrix subordinate to the l_2 norm as

$$G = \begin{bmatrix} 1 & 1 & \sqrt{6}/2 \\ \sqrt{3} & -\sqrt{3}/2 & 1/\sqrt{2} \\ 0 & 3/2 & \sqrt{6}/2 \end{bmatrix}$$

which has a condition number of $\gamma = 13.9282$. G-balancing, on the other hand, converges to $\gamma = 13.9641$, which is very close to the optimum. The pre- and post-scaling matrices derived from G-balancing are

$$\mathrm{J} = \mathrm{diag}\ (0.7035, 1.1076, 2.0659) \tag{49}$$

and

$$\mathrm{F} = \mathrm{diag}\ (0.7621, 0.3658, 0.3170) \tag{50}$$

which gives

$$\hat{G} = \begin{bmatrix} 0.5361 & 0.5146 & 0.6691 \\ 0.8441 & -0.4051 & 0.3512 \\ 0 & 0.7557 & 0.6550 \end{bmatrix} \tag{51}$$

In fact, after only two cycles of G-balancing the condition number had been reduced to 13.9728, which again demonstrates the speed with which G-balancing converges. The l_∞ norm for this example is 15.5817, showing that G-balancing has again achieved the upper bound given by theorem 1.

Conclusions

The condition number of a process model $G(s)$ is a measure of the sensitivity of the system to model uncertainties from the point of view of closed-loop stability. The condition number therefore, provides an index on the basis of which the inherent controllability of alternative process designs can be assessed. However, the condition number is dependent on the scaling of the input and output variables describing the system. If this controllability index is to be used, the input and output variables must be scaled in such a way that the true controllability properties of the process are revealed and are not hidden by numerical features of the process model.

By scaling the input and output variables such that the condition number of the model is minimized, the maximum stability robustness of the process will be revealed. The G-balancing scaling method described in this paper provides a rapid, easy to use method of scaling a system such that the minimum condition number is approached. G-balancing aims to find real diagonal scaling matrices J and F that yield a scaled matrix $\hat{G} = JGF$ in which all rows have the same magnitude (measured by the Euclidean norm), as do all columns. Such a matrix is said to be perfectly balanced and will generally have a condition number close to the minimum possible when using diagonal scaling matrices. With process models written in this form, a meaningful comparison of alternative designs is possible.

For 2×2 process models, a perfectly balanced matrix may be found analytically using a simultaneous balancing method. For systems of this order, G-balancing yields the optimal scaling matrices; that is, the scaled matrix has the minimum possible condition number subordinate to the l_2 norm. In higher order cases, repeated column and row balancing may be employed to progressively bring the matrix closer to a perfectly balanced state, although two or three balancing cycles are usually sufficient.

Notation

c_i = column in a transfer function matrix
F = input scaling matrix
$G(s)$ = process transfer function matrix
J = output scaling matrix
m = dimension of input vector $\boldsymbol{u}$
r_i = row in a transfer function matrix
s = Laplace variable
$\boldsymbol{u}$ = input vector
U, V = singular value decomposition unitary matrices
y = output vector

Greek letters

γ = process condition number
λ_i = eigenvalue
Σ = diagonal matrix of singular values
σ^* = maximum singular value
σ_* = minimum singular value
π = Perron root
$\|\cdot\|$ = vector or matrix norm

Superscripts

$\hat{}$ = scaled vector or matrix
T = complex conjugate transposition

Appendix: Proof of Theorem 3

For any Hermitian matrix, H, with eigenvalues (not necessarily distinct) $\lambda_1, \lambda_2, \ldots, \lambda_n$, it is true that

$$\mathrm{U^*HU} = \Lambda \tag{A1}$$

where

$$\Lambda = \mathrm{diag}\ (\lambda_1, \lambda_2, \ldots, \lambda_n)$$
$$U = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n]$$

and $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are the eigenvectors of H corresponding to

$\lambda_1, \lambda_2, \ldots, \lambda_n$.

Further, the eigenvectors form an orthonormal set, so that U is a unitary matrix, i.e.,

$$\mathrm{U^*U} = \mathrm{UU^*} = \mathrm{I} \tag{A2}$$

1. *Assume G has unity condition number.*

For G to have unity condition number, G^*G and GG^* must have a single eigenvalue (α say) of multiplicity n. Therefore, from Eq. A1

$$\mathrm{U^*G^*GU} = \alpha\mathrm{I} \tag{A3}$$

Premultiplying Eq. A3 by U and postmultiplying by U^*, then using Eq. A2, it follows that

$$\mathrm{G^*G} = \alpha\mathrm{I} \tag{A4}$$

Equation A4 implies that the columns of G are orthogonal, and that each column has 2-norm $\alpha^{1/2}$. To prove a similar result for the rows of G, we apply the same argument starting from

$$U^*GG^*U = \alpha I \tag{A5}$$

The above shows that a matrix with unity condition number has orthogonal rows and columns all with the same norm.

2. *Assume G has orthogonal rows and columns all with 2-norm* $\alpha^{1/2}$.

Then $G^*G = GG^* = \alpha I$.

Thus, all the singular values of G are equal to $\alpha^{1/2}$ and therefore G has a condition number of unity.

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